Anomalous scaling of a passive scalar advected by the turbulent velocity field with finite correlation time: Two-loop approximation

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The renormalization group and operator product expansion are applied to the model of a passive scalar quantity advected by the Gaussian self-similar velocity field with finite, and not small, correlation time. The inertial-range energy spectrum of the velocity is chosen in the form $E(k)\propto k^{1-2\varepsilon}$, and the correlation time at the wave number k scales as $k^{-2+\eta}$. Inertial-range anomalous scaling for the structure functions and other correlation functions emerges as a consequence of the existence in the model of composite operators with negative scaling dimensions, identified with anomalous exponents. For $\eta>\varepsilon$, these exponents are the same as in the rapid-change limit of the model; for $\eta<\varepsilon$, they are the same as in the limit of a time-independent (quenched) velocity field. For $\varepsilon=\eta$ (local turnover exponent), the anomalous exponents are nonuniversal through the dependence on a dimensionless parameter, the ratio of the velocity correlation time, and the scalar turnover time. The nonuniversality reveals itself, however, only in the second order of the ε expansion and the exponents are derived to order ε^2, including anisotropic contributions. It is shown that, for moderate order of the structure function n, and the space dimensionality d, finite correlation time enhances the intermittency in comparison with both the limits: the rapid-change and quenched ones. The situation changes when n and/or d become large enough: the correction to the rapid-change limit due to the finite correlation time is positive (that is, the anomalous scaling is suppressed), it is maximal for the quenched limit and monotonically decreases as the correlation time tends to zero.

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I. INTRODUCTION

In recent years, considerable progress has been achieved in the understanding of intermittency and anomalous scaling of fluid turbulence. The crucial role in these studies was played by a simple model of a passive scalar quantity advected by a random Gaussian field, white in time and self-similar in space, the so-called Kraichnan's rapid-change model [1]. There, for the first time the existence of anomalous scaling was established on the basis of a microscopic model [2], and the corresponding anomalous exponents were calculated within controlled approximations [3–6] and a systematic perturbation expansion in a formal small parameter [7]. Detailed review of the recent theoretical research on the passive scalar problem and the bibliography can be found in Ref. [8].

Within the approach developed in Refs. [3–6], nontrivial anomalous exponents are related to "zero modes," that is, homogeneous solutions of the closed exact differential equations satisfied by the equal-time correlation functions. In this sense, the rapid-change model appears "exactly solvable."

In a wider context, zero modes can be interpreted as statistical conservation laws of the particle dynamics [9]. The concept of statistical conservation laws appears rather general, being also confirmed in numerical simulations by Refs. [10,11], where the passive advection by the two-dimensional Navier-Stokes velocity field [10] and a shell model of a passive scalar [11] were studied. This observation is rather intriguing because in those models no closed equations for equal-time quantities can be derived due to the fact that the advecting velocity has a finite correlation time (for a passive field advected by a velocity with given statistics, closed equations can be derived only for different-time correlation functions and they involve infinite diagrammatic series).

One may thus conclude that breaking the artificial assumption of the time decorrelation of the velocity field is the crucial point [10,11].

An important issue related to the effects of the finite correlation time is the universality of the anomalous exponents. It was argued that the exponents may depend on more details of the velocity statistics than only the exponents η and ε [12]. This idea was supported in Refs. [13,14], where the case of short but finite correlation time was considered for the special case of a local turnover exponent. In those studies, the anomalous exponents were derived to first order in small correlation time, with Kraichnan's rapid-change model [13] or analogous shell model for a scalar field [14] taken as zeroth-order approximations. The exponents obtained appear nonuniversal through the dependence on the correlation time.

In Ref. [7] and subsequent papers [15–19], the field theoretic renormalization group (RG) and operator product expansion (OPE) were applied to the rapid-change model and its descendants. In that approach, anomalous scaling emerges as a consequence of the existence in the model of composite operators with negative scaling dimensions, identified with the anomalous exponents. This allows one to construct a systematic perturbation expansion for the anomalous exponents, analogous to the famous ε expansion in the RG theory of critical behavior, and to calculate the exponents to the second [7,15] and third [16] orders. For passively advected vector fields, where the calculations become rather involved, the exponents for higher-order correlation functions were de-

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rived by means of the RG techniques to the leading order in ε in Refs. [17].

Besides the calculational efficiency, an important advantage of the RG approach is its relative universality: it is not related to the aforementioned solvability of the rapid-change model and can also be applied to the case of finite correlation time or non-Gaussian advecting field. In Ref. [18] (see also Ref. [19] for the case of compressible flow) the RG and OPE were applied to the problem of a passive scalar advected by a Gaussian self-similar velocity with finite (and not small) correlation time. The energy spectrum of the velocity in the inertial range has the form $E(k) \propto k^{1-2\varepsilon}$, while the correlation time at the wave number k scales as $k^{-2+\eta}$. It was shown that, depending on the values of the exponents ε and η, the model reveals various types of inertial-range scaling regimes with nontrivial anomalous exponents. For $\eta > \varepsilon$, they coincide with the exponents of the rapid-change model and depend on the only parameter $2\varepsilon - \eta$, while for $\varepsilon > \eta$ they coincide with the exponents of the opposite (“quenched” or “frozen”) case and depend only on ε.

The most interesting case is $\eta = \varepsilon$, when the exponents can be nonuniversal through the dependence on the correlation time (more precisely, on the ratio u of the velocity correlation time and the turnover time of the passive scalar). In the field theoretic language, the nonuniversality of the exponents in this regime is a consequence of the degeneracy of the corresponding fixed point of the RG equations. It agrees with the findings of Refs. [13,14] since the borderline $\eta = \varepsilon$, including the “Kolmogorov” point $\eta = \varepsilon = 4/3$, corresponds to the case of a local turnover exponent. It is also interesting to note that the same relation $\eta = \varepsilon$ for the boundary between the time-decorrelated and quenched cases is encountered in a model of passive advection by a strongly anisotropic flow, studied in Refs. [20]. It was argued in Ref. [21] that the same boundary will be observed with very general assumptions on the velocity statistics. Although the possibility of the nonuniversality of anomalous exponents for $\eta = \varepsilon$ was demonstrated by the rigorous RG analysis, the practical calculation by Ref. [18] has shown that they appear universal (independent of u) to the first order in ε and η: in the one-loop approximation, the anomalous dimensions of the relevant composite operators depend on a combination of the model parameters (couplings) that remains constant along the line of the fixed points. This fact is rather disappointing because it means that in the one-loop approximation it is impossible to judge how the finite correlation time affects intermittency, in particular, whether the anomalous scaling is enhanced or suppressed in comparison with the rapid-change or quenched limits.

In this paper, we present the anomalous exponents to order $O(\varepsilon^2)$ (two-loop approximation) for the most interesting case $\eta = \varepsilon$, including the exponents of the anisotropic contributions, and study their dependence on u. [It is not necessary to separately consider the cases $\eta > \varepsilon$ ($\eta < \varepsilon$), because the corresponding exponents are the same as for the rapid-change (quenched) velocity field and can be obtained from the case $\eta = \varepsilon$ in the limits $u \to \infty$ ($u \to 0$)].

In Sec. II, we describe our model and its interesting special cases. In Sec. III, we briefly recall the field theoretic formulation, the RG and OPE approach to the model, and the $O(\varepsilon)$ result for the anomalous exponents [18]. The results of the two-loop calculation are presented and discussed in Sec. IV: in Sec. IV A, we give the anomalous exponents to order $O(\varepsilon^2)$ and then discuss them separately for the isotropic (Sec. IV B) and anisotropic (Sec. IV C) contributions.

The main conclusion of the paper can be formulated as follows: the qualitative effect of the finite correlation time on the anomalous scaling depends essentially on the correlation function considered, the value of u, and the space dimensionality d. For the low-order structure functions and in low dimensions ($d = 2$ or 3), the inclusion of finite correlation time enhances the intermittency in comparison with both the limits: the time-decorrelated ($u = \infty$) and time-independent ($u = 0$) ones. Although the anomalous exponents have well-defined limits for $u \to 0$, they show interesting irregularities in the vicinity of the quenched limit: a rapid falloff when $u = 0$ increases from zero, with infinite slope for $d = 2$, with a pronounced minimum for $u \sim 1$. On the contrary, the behavior in the region of large u is smooth, like for the shell model studied in Ref. [14]. For higher-order structure functions and large d, the anomalous scaling is always weaker in comparison with the rapid-change limit and the corresponding (positive) correction is maximal for $u = 0$ and monotonically decreases to zero as u tends to infinity.

II. THE MODEL

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ is described by the stochastic equation

$$\nabla_t \theta = \nu_0 \partial^2 \theta + f, \quad \nabla_t \equiv \partial_t + v_i \partial_i, \tag{2.1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, ∂^2 is the Laplace operator, $\mathbf{v}(x) \equiv \{v_i(x)\}$ is the divergence-free (owing to the incompressibility) velocity field, and $f \equiv f(x)$ is an artificial Gaussian random noise with zero mean and correlation function

$$\langle f(x) f(x') \rangle = C(t-t', \mathbf{r}), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'. \tag{2.2}$$

The form of the correlator is unessential; it is only important that the function C in Eq. (2.2) decreases rapidly for $r \gg L$, where L is some integral scale. The noise maintains the steady state of the system and, if C depends on the vector $\mathbf{r}$ and not only its modulus $r \equiv \mathbf{r}$, is a source of large-scale anisotropy. In a more realistic formulation, the noise is replaced by an imposed constant gradient of the scalar field; see, e.g., Refs. [5,6,18,19,22].

In the real problem, the velocity field satisfies the Navier-Stokes equation. Following Refs. [14,18,19,22], we assume for $\mathbf{v}(x)$ in Eq. (2.1) a Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x') \rangle = \int \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) D_v(t-t', k) \times \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')], \tag{2.3}$$

where $P_{ij}(\mathbf{k})\equiv\delta_{ij}-k_ik_j/k^2$ is the transverse projector and the function $D_v(t,k)$ will be chosen in the form

$$D_v(t-t',k)=\frac{D_0}{2u_0}\,\frac{1}{k^{d-2+2\varepsilon}}\exp[-\omega_k(t-t')],$$

$$\omega_k=u_0\nu_0\;k^{2-\eta}. \tag{2.4}$$

Here D_0 and u_0 are positive amplitude factors and the positive exponents ε and η play the part of small expansion parameters in the RG theory; see Refs. [18,19]. It is also convenient to introduce the "coupling constant" $g_0\equiv D_0/\nu_0^2$ (expansion parameter in the ordinary perturbation theory). The infrared (IR) regularization is provided by the sharp cutoff in all momentum integrals from below at $k=m$ with $m\sim 1/L$.

As was pointed out in Ref. [22], the Gaussian model (2.3), (2.4) suffers from the lack of Galilean invariance and therefore does not take into account the self-advection of turbulent eddies. It is well known that the different-time correlations of the Eulerian velocity field are not self-similar, as a result of these "sweeping effects," and depend substantially on the integral scale; see, e.g., Ref. [23]. It would be much more appropriate to use Eqs. (2.3) and (2.4) in the Lagrangian frame, but this is embarrassing due to the daunting task of relating Eulerian and Lagrangian statistics for a flow with a finite correlation time (which is not a problem for the zero correlation time limit). However, the results of Ref. [22] show that the Gaussian model gives reasonable description of the passive advection in an appropriate frame, where the mean velocity field vanishes. To justify the model (2.3), (2.4), we also note that we shall be interested preferably in the equal-time, Galilean invariant quantities (structure functions), which are not affected by the sweeping, and we expect that their absence in the Gaussian model is not crucial.

The model (2.3), (2.4) contains two special cases that possess some interest on their own: in the limit $u_0\to\infty$, $g_0'\equiv g_0/u_0^2=$const we arrive at the rapid-change model,

$$D_v(\omega,k)\to g_0'\nu_0\;\delta(t-t')k^{-d-2\varepsilon+\eta}, \tag{2.5}$$

while the limit $u_0\to 0$, $g_0''\equiv g_0/2u_0=$const corresponds to the case of a quenched (time-independent) velocity field,

$$D_v(\omega,k)\to g_0''\nu_0^2\;k^{-d+2-2\varepsilon}. \tag{2.6}$$

The latter case has a close formal resemblance with the well-known models of random walks in random environment with long-range correlations; see Refs. [24,25].

III. RENORMALIZATION GROUP AND OPERATOR PRODUCT EXPANSION

The RG theory of the model (2.1)–(2.4) is presented in Refs. [18,19] in detail; below we briefly recall only the necessary information. The stochastic problem (2.1)–(2.4) can be cast as a field theory with action functional

$$S(\theta,\theta',v)=-\mathbf{v}D_v^{-1}\mathbf{v}/2+\theta' D_f\theta'/2+\theta'[-\boldsymbol{\nabla}_t+\nu_0\partial^2]\theta, \tag{3.1}$$

where θ' is an auxiliary scalar field and D_f and D_v are correlators (2.2) and (2.3), respectively. In Eq. (3.1), all the required integrations over $x=(t,\mathbf{x})$ and summations over the vector indices are understood.

The model (3.1) is logarithmic for $\varepsilon=\eta=0$; the ultraviolet (UV) singularities have the form of poles in various linear combinations of ε and η in the correlation functions. They can be removed by the only counterterm of the form $\theta'\partial^2\theta$, which is equivalent to the following multiplicative renormalization of the parameters g_0, u_0, and ν_0 in the action functional (3.1):

$$\nu_0=\nu Z_\nu,\quad g_0=g\mu^{2\varepsilon+\eta}\;Z_g,\quad u_0=u\mu^{\eta}\;Z_u, \tag{3.2}$$

where g, u, and ν are the renormalized counterparts of the bare parameters; μ is the reference mass in the minimal subtraction (MS) scheme, which we always use in practical calculations; and $Z_i=Z_i(g,u;d;\varepsilon,\eta)$ are the renormalization constants satisfying the identities

$$Z_g=Z_\nu^{-3},\quad Z_u=Z_\nu^{-1}. \tag{3.3}$$

Fixed points of the corresponding RG equations are found from the requirement that the β functions,

$$\beta_g\equiv\widetilde{\mathcal{D}}_\mu g=g[-2\varepsilon-\eta+3\gamma_\nu],$$

$$\beta_u\equiv\widetilde{\mathcal{D}}_\mu u=u[-\eta+\gamma_\nu],\quad \gamma_\nu\equiv\widetilde{\mathcal{D}}_\mu\ln Z_\nu \tag{3.4}$$

vanish. Here $\widetilde{\mathcal{D}}_\mu$ is the operation $\mu\partial_\mu$ for fixed bare parameters and the relations between β functions and the anomalous dimension γ_ν result from the definitions and the relation (3.3).

The exact relation $\beta_g/g-3\beta_u/u=2(\eta-\varepsilon)$, following from Eq. (3.4), shows that the β functions cannot vanish simultaneously for finite values of their arguments, except for the case $\eta=\varepsilon$. Therefore, to find the fixed points we must set either $u=\infty$ or $u=0$ and simultaneously rescale g so that the anomalous dimension γ_ν remain finite. These two options correspond to the two limits (2.5) and (2.6), so that the rapid-change and quenched cases are fixed points of the general model. The analysis shows that the former is IR stable (and thus describes the inertial-range asymptotic behavior) for $\eta>\varepsilon$, while the latter is IR stable for $\eta<\varepsilon$.

The most interesting case is $\eta=\varepsilon$, when the β functions become proportional and the set $\beta_g=\beta_u=0$ reduces to a single equation. As a result, the corresponding fixed point is degenerate: rather than a point, one obtains a line of fixed points in the g-u plane. They can be labeled by the value of the parameter u, which has the meaning of the ratio of the velocity correlation time and the scalar turnover time.

Existence of the IR stable fixed points implies certain scaling properties of various correlation functions at scales larger than the dissipative length $\sim g_0^{-1/3\varepsilon}$. In particular, for the equal-time structure functions

$$\mathcal{S}_n(\mathbf{r})=\langle[\theta(t,\mathbf{x})-\theta(t,\mathbf{x}')]^n\rangle,\quad \mathbf{r}=\mathbf{x}-\mathbf{x}', \tag{3.5}$$

one obtains

$$\mathcal{S}_n(\mathbf{r}) = D_0^{-n/2} r^{n(1-\varepsilon/2)} F_n(m\mathbf{r}) \tag{3.6}$$

(odd structure functions are nontrivial if the correlation function $\langle v f\rangle$ is nonzero or if a constant gradient of the scalar field is imposed). In the presence of anisotropy the scaling functions $F_n(m\mathbf{r})$ can be decomposed into irreducible representations of the SO(d) group. In the simplest case of uniaxial anisotropy (which is sufficient to reveal *all* anomalous exponents) one can write

$$F_n(m\mathbf{r}) = P_l(z) F_{nl}(mr), \quad z = (\mathbf{n}\cdot\mathbf{r})/r, \tag{3.7}$$

where $P_l(z)$ is the lth order Gegenbauer polynomial (Legendre polynomial for $d=3$) and $\mathbf{n}$ is a unit vector that determines the distinguished direction.

The leading behavior of the functions F_{nl} for $mr \ll 1$ (inertial range) is found from the corresponding operator product expansion and has the form

$$F_{nl} \propto (mr)^{\Delta_{nl}}, \tag{3.8}$$

where the "anomalous exponent" Δ_{nl} is nothing other than the critical dimension of the irreducible traceless lth rank tensor composite operator built of n fields θ and minimal possible number of derivatives [18]. For $l \leqslant n$ such an operator has the form

$$\partial_{i_1}\theta \cdots \partial_{i_l}\theta \ (\partial_i\theta\partial_i\theta)^p + \cdots, \quad n = l + 2p. \tag{3.9}$$

Here the dots stand for the appropriate subtractions involving the Kronecker δ symbols, which ensure that the resulting expressions are traceless with respect to any given pair of indices, for example, $\partial_i\theta\partial_j\theta - \delta_{ij}\partial_k\theta\partial_k\theta/d$. We also note that the numbers n and l necessarily have the same parity, that is, they can only be simultaneously even or odd.

For the most interesting case of the degenerate fixed point, the dimensions Δ_{nl} are calculated in the form of series in the only independent exponent $\varepsilon = \eta$, that is,

$$\Delta_{nl} = \sum_{k=1}^{\infty} \varepsilon^k \ \Delta_{nl}^{(k)}. \tag{3.10}$$

In the lowest order one obtains [18]

$$\Delta_{nl}^{(1)} = \frac{-n(n-2)(d-1) + \lambda_l(d+1)}{2(d-1)(d+2)} \tag{3.11}$$

with $\lambda_l \equiv l(d+l-2)$. For $k \geqslant 2$, the coefficients $\Delta_{nl}^{(k)}$ depend not only on d but also on the parameter u, the ratio of the velocity correlation time, and the scalar turnover time, which labels fixed points in the g-u plane (see above).

The reader not interested in the details of practical calculation can skip the end of this section and pass to the result for $\Delta_{nl}^{(2)}$. Calculation of the higher-order coefficients in the ε expansions for the rapid-change model is presented in Refs. [15,16] in detail. Analogous calculations for the finite correlated case are more difficult in two respects. First, there are more relevant Feynman diagrams in the same order of perturbation theory (for zero correlation time, many diagrams contain closed circuits of retarded propagators and vanish). Second, and the more important distinction, is that the diagrams for the finite correlated case involve *two* different dispersion laws: $\omega \propto k^2$ for the scalar and $\omega \propto k^{2-\eta}$ for the velocity fields. As a result, the calculation, as well as expressions for the renormalization constants, become rather cumbersome already in the lowest (one-loop) approximation; see Refs. [18,19].

The latter difficulty can be circumvented as follows. Careful analysis shows that in the MS scheme all the needed anomalous dimensions, γ_ν from Eq. (3.4) and $\gamma_{nl} \equiv \widetilde{\mathcal{D}}_\mu \ln Z_{nl}$, in contrast to the respective renormalization constants Z_ν and Z_{nl}, are independent of the exponents ε and η in the two-loop approximation (for the one-loop approximation this is obvious from the explicit expressions; see Refs. [18,19]). It is thus sufficient to calculate them for any specific choice of the exponents ε and η that guarantees UV finiteness of the diagrams. The most convenient choice is $\eta = 0$ and arbitrary ε: all the diagrams remain finite, the exponents in the aforementioned dispersion laws become identical, and the practical calculations drastically simplify and become feasible.

To avoid possible misunderstandings, it should be emphasized that such an independence is *not* guaranteed by the renormalizability of the model. The renormalizability in the analytic regularization only guarantees that the renormalization scheme can be chosen such that the correlation functions, along with the coefficients β and γ in the RG equations, will be analytic at the origin in the space of two complex variables ε and η [26]. We used another scheme in which the functions β and γ are *independent* of ε and η in the first two orders, which does not exclude *nonanalytic* dependence on these parameters in higher orders. We expect that in the three-loop approximation nonanalytic constructions such as $(\varepsilon+\eta)/(\varepsilon+2\eta)$ will indeed appear in the anomalous dimensions, in particular, due to the necessity to take into account UV finite parts of the two-loop diagrams (with our choice of the sharp IR cutoff in Eq. (2.4), the one-loop diagrams have no UV finite parts; cf. [16] for the rapid-change case).

Thus, we conclude that the knowledge of the renormalization constants for $\eta = 0$ is sufficient to obtain the anomalous dimensions, β functions, coordinates of the fixed points, and the critical dimensions of composite operators for arbitrary values of η and ε, including the most interesting case $\eta = \varepsilon$, which we always discuss from now on.

IV. ANOMALOUS EXPONENTS IN THE TWO-LOOP APPROXIMATION

A. General expressions

We have performed the complete two-loop calculation of the RG functions (3.4) and the critical dimensions (3.10) of the composite operators (3.9) for arbitrary values of n, l, d, and u and obtained the following expression for the second coefficient in expansion (3.10):

$$\Delta_{nl}^{(2)}=\frac{1}{(d-1)^2(d+2)^2(d+4)}\times(2(d+4)\mathcal{A}[n(n-2)(d-1)+\lambda_l]+(n-2)\{6\mathcal{B}[n(n-4)(d-1)+3\lambda_l]+9\mathcal{C}[n(d+n)(d-1)-\lambda_l(d+1)]\}) \tag{4.1}$$

with $\lambda_l\equiv l(d+l-2)$. Here and below we denote

$$\mathcal{A}=\frac{(u-1-1/u)(d+1)}{2(d+2)(1-u)}+\frac{(d+1)}{2(d+4)(1-u)u(1+u)^2}F_3\left(\frac{1}{(u+1)^2}\right)+\frac{2ud(d+2)}{(1-d)(1-u)}\mathcal{J}(u,d), \tag{4.2a}$$

$$\mathcal{B}=\frac{(d+1)}{3(1-u)^2(d+4)}\left[\frac{u}{u+1}F_3\left(\frac{1}{2(u+1)}\right)-\frac{1}{(u+1)^2}F_3\left(\frac{1}{(u+1)^2}\right)-\frac{u^2}{4}F_3\left(\frac{1}{4}\right)\right], \tag{4.2b}$$

$$\mathcal{C}=\frac{1}{9(1-u)^2}\left\{\frac{3u^2(d-1)}{4}F_2\left(\frac{1}{4}\right)-\frac{u[2d-1+u(d-2)]}{(u+1)}F_2\left(\frac{1}{2(u+1)}\right)+\frac{[d+1+2u(d-2)]}{(u+1)^2}F_2\left(\frac{1}{(u+1)^2}\right)-\frac{u^2(d+1)}{(d+4)}F_3\left(\frac{1}{4}\right)+\frac{4u(d+1)}{(u+1)(d+4)}F_3\left(\frac{1}{2(u+1)}\right)-\frac{4(d+1)}{(u+1)^2(d+4)}F_3\left(\frac{1}{(u+1)^2}\right)\right\}. \tag{4.2c}$$

We also have denoted $F_k(x)\equiv F(1,1;d/2+k;x)$ for the hypergeometric series

$$F(a,b;c;z)\equiv 1+\frac{ab}{c}\ z+\frac{a(a+1)b(b+1)}{c(c+1)}\left(\frac{z^2}{2!}\right)+\cdots.$$

The values of F_k entering into Eq. (4.2) can be related by the recurrent relation

$$(x-1)F_2(x)=x(d+2)F_3(x)/(d+4)-1,$$

but the resulting expressions look more cumbersome and we shall keep both F_2 and F_3 in the formulas.

The quantity $\mathcal{J}(u,d)$ in Eq. (4.2a) can only be expressed in the form of a single convergent integral, suitable for numerical calculation,

$$\mathcal{J}(u,d)=\frac{\Gamma(d/2)}{\sqrt{\pi}\ \Gamma[(d-1)/2]}\int_0^1 dz\ \frac{(1-z^2)^{d/2}}{(u-1)^2+4uz^2}\times\left\{z^2(1-z^2)\ln\left(\frac{1+u}{2}\right)-z(u-1+2z^2)\arcsin z-\frac{z(1-z^2)^{1/2}(1-u-z^2)}{[2(1+u)-z^2]^{1/2}}\times\arctan\left[\frac{z[2(1+u)-z^2]^{1/2}}{(1+u-z^2)}\right]\right\}, \tag{4.3}$$

where $\Gamma(\cdots)$ is the Euler γ function.

The quantities (4.2), and hence the dimensions (4.1), have finite limits for $u\to\infty$ and $u\to 0$. In the first limit, $\Delta_{nl}^{(2)}$ coincides with the known result for the Kraichnan's rapid-change model (see Ref. [7] for $l=0$ and 2 and Ref. [15] for general l). The $O(1/u)$ correction to the rapid-change limit can be found from the following asymptotic expressions for the coefficients (4.2):

$$\mathcal{A}=\frac{(d+1)}{2(d+2)}\ (1+2/u)+O(1/u^2), \tag{4.4a}$$

$$\mathcal{B}=\frac{(d+1)}{12(d+4)}F_3\left(\frac{1}{4}\right)\ (1+2/u)+O(1/u^2), \tag{4.4b}$$

$$\mathcal{C}=\left[-\frac{(d-1)}{12}F_2\left(\frac{1}{4}\right)+\frac{(d+1)}{9(d+4)}F_3\left(\frac{1}{4}\right)\right]+\frac{1}{u}\left[-\frac{(d-1)}{6}F_2\left(\frac{1}{4}\right)+\frac{2(d+1)}{9(d+4)}\times F_3\left(\frac{1}{4}\right)+\frac{(d-2)}{9}\right]+O(1/u^2). \tag{4.4c}$$

The opposite case, $u=0$, corresponds to the quenched (time-independent) velocity field. This case was extensively studied in connection with the so-called "random-random walks" (random walks in random environments); see the review paper [24] and references therein. Our results for the function β_g from Eq. (3.4) and the corresponding fixed point are in agreement with the two-loop results quoted in Ref. [24] for the model of random-random walks. To our knowledge, the dimensions of composite operators (3.9) have not been studied in that context, and below we give the asymptotic expressions for the coefficients (4.2) sufficient to find the dimensions (4.1) up to order $O(u)$ near the quenched limit:

$$\mathcal{A}=-\frac{(d+1)(3d+4)}{2d(d+2)}+u(d+1)\left\{\frac{(3d+2)}{2d(d-2)}-\frac{2}{d(d+2)}+\frac{\ln 2}{(d+4)}+\frac{(d+3)}{(d+4)(d+6)}F_4\left(\frac{1}{2}\right)-\frac{2(d+3)}{(d+4)^2}\right\}, \quad (4.5a)$$

$$\mathcal{B}=-\frac{(d+1)}{3(d+2)}+\frac{u(d+1)}{3}\left\{\frac{1}{(d+4)}F_3\left(\frac{1}{2}\right)+\frac{4}{d(d+2)}\right\}, \quad (4.5b)$$

$$\mathcal{C}=\frac{(d+1)(d^2+4)}{9d(d+2)}+\frac{u}{9}\left\{(1-2d)F_2\left(\frac{1}{2}\right)+\frac{4(d+1)}{(d+4)}F_3\left(\frac{1}{2}\right)-\frac{4(d+1)(d+2)}{d(d-2)}+\frac{2(d^2+2d+4)}{(d+2)}\right\}, \quad (4.5c)$$

up to corrections of order $O(u^2)$.

It is worth noting that the $O(u)$ terms in Eqs. (4.5a) and (4.5c) contain poles in $(d-2)$ and thus diverge for $d=2$. Analysis shows that, for $d=2$, the leading correction to the result for $u=0$ is not analytical in u and has the form $u\ln u$. Formally, the singularity at $d=2$ is explained as follows. Some of the two-loop diagrams contain "energy denominators" of the form $(\mathbf{k}+\mathbf{q})^2+O(u)$, where $\mathbf{k}$ and $\mathbf{q}$ are two independent integration momenta. The numerators contain factors $\propto[P_{ij}(\mathbf{k})q_iq_j]^2$ stemming from transverse projectors in the propagators. These factors suppress the singularity at $\mathbf{k}=-\mathbf{q}$, occurring in the denominators for $u=0$, and ensure the existence of the integrals over $\mathbf{k}$ and $\mathbf{q}$. However, the "collinear" divergence at $\mathbf{k}=-\mathbf{q}$ occurs if the $O(u)$ correction to the denominators is taken into account. Physically, this divergence can be related to a strong resonant interaction between the excitations of the passive scalar field with the opposite momenta $\mathbf{k}=-\mathbf{q}$ of equal moduli in two dimensions. We shall see below that this singularity remarkably affects the behavior of the dimensions (4.1) for the values of d much larger than $d=2$.

Many studies have been devoted to the analysis of the inertial-range turbulence in the limit $d\to\infty$ [4,27–29]. Our model has no finite "upper critical dimension," above which anomalous scaling would vanish. Like in the rapid-change case [27] and, probably in the Navier-Stokes turbulence [28,29], the anomalous scaling disappears at $d=\infty$, but it reveals itself already in the $O(1/d)$ approximation. Along with the results [4] for the scalar rapid-change model, where the $O(1/d)$ expression for the anomalous exponents were derived for any ε, this confirms the importance of the large-d expansion for the issue of anomalous scaling in fully developed turbulence.

Straightforward analysis of the expressions (4.2) shows that, for $d\to\infty$, one has $\mathcal{A}=O(d^0)$ [it is important here that $\mathcal{J}(u,d)=O(1/d)$], $\mathcal{B}=O(d^0)$, and $\mathcal{C}=O(d)$, namely,

$$\mathcal{C}=\frac{(u+2)(3u+2)}{36(u+1)^2}\,d+O(d^0). \quad (4.6)$$

It then follows that for large d, the dimension (4.1) behaves as $O(1/d^2)$ and is completely determined by the only contribution with the coefficient $\mathcal{C}$. This gives

$$\Delta_{nl}^{(2)}=\frac{(n-2)(n-l)(u+2)(3u+2)}{4(u+1)^2d^2}+O(1/d^3). \quad (4.7)$$

The general expressions (4.1), (4.2) are rather cumbersome, and in the subsequent sections we shall separately discuss isotropic contribution (even n, $l=0$) and anisotropic ones (general n, $l\neq 0$).

B. Isotropic sectors

Expression (4.1) simplifies for the most important case of the isotropic sector (even n and $l=0$),

$$\Delta_{n0}^{(2)}=\frac{n(n-2)}{(d-1)(d+2)^2(d+4)}\times\{2(d+4)\mathcal{A}+6(n-4)\mathcal{B}+9(d+n)\mathcal{C}\}. \quad (4.8)$$

Equation (4.8) gives $\Delta_{20}^{(2)}=0$ in agreement with the exact result $\Delta_{20}=0$ [18]. This means that the second-order structure function is not anomalous. The formal proof is based on certain Schwinger equation, which has the meaning of the energy conservation law; it is almost identical to the analogous proof for the Kraichnan model, given in Ref. [7]. In the zero-mode approach to the Kraichnan model, the absence of anomaly for the second-order correlation function can be related to the fact that for the isotropic sector there is no nontrivial geometry in configurations of two particles: everything is defined by the distance between them and no zero mode can thus exist; see, e.g., Ref. [8].

For the simplest nontrivial case $n=4$, one obtains

$$\Delta_{40}^{(2)}=8(2\mathcal{A}+9\mathcal{C})/(d-1)(d+2)^2, \quad (4.9)$$

that is, the quantity $\mathcal{B}$ does not enter into the result. For $n\geqslant 6$, all the coefficients (4.2) contribute to the result.

In Fig. 1, we show the behavior of the quantity

$$\zeta_n\equiv[\Delta_{n0}^{(2)}-\Delta_{n0}^{(2)}|_{u=\infty}]/n^3 \quad (4.10)$$

for $n=4$, 6, 8, and 20 (from below to above) as a function of u for several values of d. We have subtracted the value of the dimension for the rapid-change case, such that the curves approach zero as $u\to\infty$, and divided the difference by n^3, such that the results for different n's have the same order of magnitude [the quantity (4.1) is a third-order polynomial in n]. It is worth noting that, since the leading coefficient (3.11) is independent of u, it drops from the difference $\Delta_{n0}-\Delta_{n0}|_{u=\infty}$ of the *exact* dimensions, and in the leading order $O(\varepsilon^2)$ the latter is proportional to the quantity ζ_n introduced above.

As one can easily see from Fig. 1, the qualitative behavior of ζ_n depends essentially on the values of n and d. For moderate n and d (e.g., $n=4$, 6, and 8 for $d=2$ and 3), finite correlation time enhances the intermittency (anomalous dimensions become more negative) in comparison with both

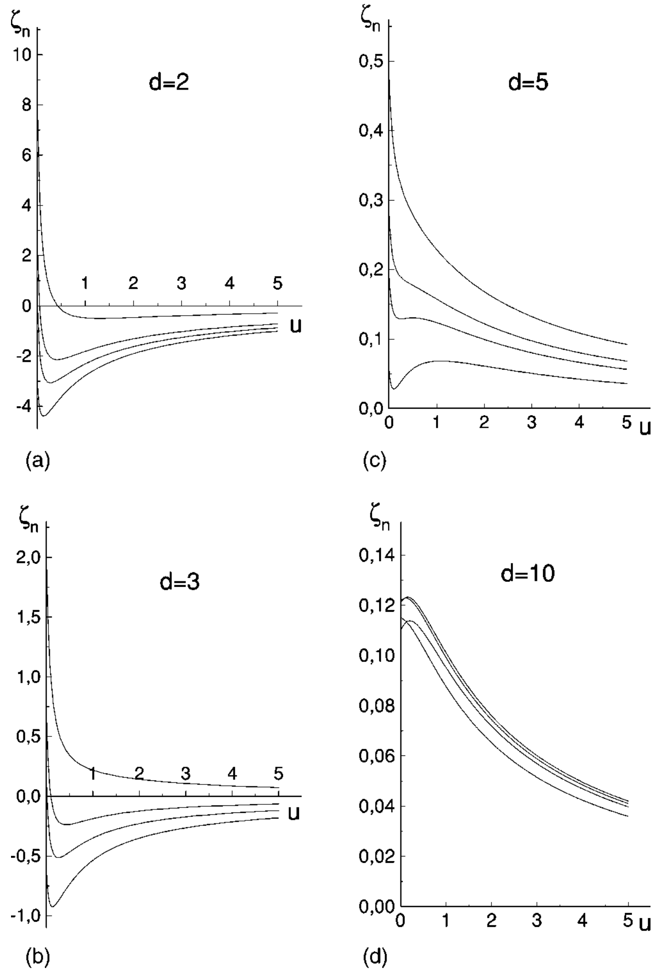

FIG. 1. Behavior of the quantity ζ_n from Eq. (4.10) for $n=4$, 6, 8, and 20 (from below to above) as a function of u for $d=2$, 3, 5, and 10 (from the left to the right) in the units of 10^{-3}.

the limits: the rapid-change ($u=\infty$) and quenched ($u=0$) ones. While the $O(1/u)$ correction leads to a smooth decrease of ζ_n for increasing $1/u$ from zero (in agreement with the numerical simulation of Ref. [14] for a shell model near the rapid-change limit), the rapid falloff of ζ_n is observed for increasing u from the quenched limit $u=0$. As a result of the competition between these two effects, ζ_n is not a monotonous function of u and has a pronounced minimum in the interval between 0 and 1.

We recall that the slopes of the functions ζ_n at $u=0$ are infinite in two dimensions for all values of n due to the presence of poles $(d-2)$ in the $O(u)$ terms in Eqs. (4.5a) and (4.5c); see Sec. IV A. For $d>2$, the slopes become finite but they still remain very steep for $d=3$ and lead to a rapid falloff of ζ_n, as Fig. 1(b) shows. This fact also suggests that the quenched case, in contrast with the rapid-change one, can hardly serve as a good zero-order approximation in studying more realistic models of passive advection by the velocity field with finite correlation time.

If n or d is large enough, the minimum becomes less pronounced, the behavior of ζ_n becomes more regular (e.g., $n=20$ for $d=2$) and eventually ζ_n becomes a monotonically decreasing function of u ($n=20$ for $d=3$). For such values of n or d, the function ζ_n approaches zero as $u\to\infty$ from above. In other words, the $O(1/u)$ correction to the rapid-change limit suppresses the intermittency, in contrast with the case of moderate n.

In the limit of large d, from Eq. (4.7) one easily obtains

$$\Delta_{n0}^{(2)}-\Delta_{n0}^{(2)}|_{u=\infty}=\frac{n(n-2)(2u+1)}{4(u+1)^2d^2}+O(1/d^3) \quad (4.11)$$

[note that the $O(n^3)$ term in this approximation disappears and $\Delta_{n0}^{(2)}$ becomes only quadratic in n]. One can see that for all values of n, the difference (4.11) is positive, decreases monotonically when u grows, and approaches zero from above when $u\to\infty$.

It should be kept in mind, however, that any conclusion about the large-n behavior of the exponents, based on a finite-order approximation of the ε expansion, can be trusted only if ε is small enough, namely $\varepsilon n \ll 1$. Like for the rapid-change case (see, e.g., the discussion in Ref. [7]), higher-order terms of the ε expansion (3.10) contain additional powers of n and the actual expansion parameter appears to be $n\varepsilon$ rather than ε itself. Thus the correct analysis of the large-n behavior requires resummation of the ε series with the additional condition that $n\varepsilon \simeq 1$, but we know of no model in which such a resummation has been performed.

On the contrary, the analysis of the Feynman diagrams shows that the coefficients in Eq. (3.10) are expandable in $1/d$ and thus the large-d behavior of the exponents is still in the realm of application of the ε expansion. What is more, for the rapid-change case, the terms of order ε^2 and higher in the dimension Δ_{n0} behave as $O(1/d^2)$ for large d and therefore, has no contribution in the $O(1/d)$ approximation; as a result, the first order of the $1/d$ expansion for Δ_{n0} is contained completely in the first order of its ε expansion; see Ref. [4]. Our result (4.11) suggests that this is equally true for the case of a finite correlation time.

In Ref. [13], the $O(1/u)$ correction to the rapid-change case was derived by the zero-mode techniques in the limit of large d and arbitrary (not small) values of ε, for the case of a local turnover exponent ($\varepsilon=\eta$). Although the anomalous exponents were shown to be nonuniversal (dependent on u), our results disagree with Ref. [13] in two respects. First, due to the universality (independence of u) of the leading term (3.11), the ratio (4.11) is of order $O(\varepsilon^2)$ and not $O(\varepsilon)$. Second, the $O(1/u)$ correction in Eq. (4.11) is positive for all n, while, according to Ref. [13], inclusion of the finite correlation time makes the anomalous exponents more negative in comparison with the rapid-change limit for all n and ε. It is not clear whether this disagreement can be explained by some distinctions between our model (2.3), (2.4) and the velocity ensemble employed in Ref. [13]. It is possible to show, however, that any modification of the function (2.4) consistent with the RG analysis performed in Ref. [18] and Sec. III above leads to a universal (independent of u) expression for the leading term in $\Delta_{nl}^{(1)}$, so that the $O(1/u)$ correction to the zero-correlated limit remains of order $O(\varepsilon^2)$.

The changeover from the behavior typical to low spatial dimensions to the behavior described by Eq. (4.11) also produces interesting patterns, as illustrated by Fig. 1 for $d=5$ and 10.

C. Anisotropic sectors

Let us turn to the analysis of anisotropic contributions in the structure functions (3.6), (3.7), described by the dimensions (3.10) with $l\neq 0$. We recall that such contributions appear in the inertial-range expression (3.7) if the forcing Eq. (2.2) is chosen to be anisotropic, or a constant gradient of the scalar field is imposed.

An important property of the first-order result (3.11) is that for any fixed n, the quantity $\Delta_{nl}^{(1)}$ increases monotonically with l [15]. One can say that the exponents, associated with tensor composite operators (3.9), exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank l, the less is the dimension and, consequently, the more important is the corresponding contribution to the inertial-range expression (3.7). The leading terms in the even structure functions (3.5) are given by the scalar operators (3.9) with $l=0$, that is, they are the same as in the model with isotropic forcing (we recall that n and l should be simultaneously even or odd).

This behavior is in agreement with the existing phenomenological ideas, according to which the anisotropy introduced at large scales by the forcing (boundary conditions, geometry of an obstacle etc.) dies out when the energy is transferred down to smaller scales owing to the cascade mechanism [30]. The hierarchy of anisotropic contributions appears rather universal, being also observed for a vector (magnetic) field, advected by the Kraichnan velocity ensemble [31]; for the scalar advected by the two-dimensional Navier-Stokes velocity field [32] and for the turbulent velocity field itself [33].

Nevertheless, the anisotropy survives in the inertial range and reveals itself in dimensionless ratios involving *odd* structure functions,

$$\mathcal{R}_k\equiv\mathcal{S}_{2k+1}/\mathcal{S}_2^{k+1/2}. \tag{4.12}$$

For a number of models it was shown that the skewness factor $\mathcal{R}_1$ decreases down the scales but slower than predicted by phenomenological theories [5,6], while the higher-order odd ratios (hyperskewness $\mathcal{R}_2$ etc.) increase, thus signaling persistent small-scale anisotropy [18,19,31,32]. Due to the aforementioned hierarchy of the dimensions (3.11), the leading terms in the odd structure functions (3.5) in our model are determined by the vector operators (3.9) with $l=1$, and it is easy to check the above statements from the explicit expression (3.11).

Of course, the $O(\varepsilon^2)$ contribution (4.1) cannot change all the above properties of the dimensions Δ_{nl}, determined by their leading $O(\varepsilon)$ contribution (3.11), as far as small values of ε are concerned. However, since the dependence on u occurs only in the $O(\varepsilon^2)$ contribution, it should be taken into account if one wishes to discuss how finite correlation time affects the hierarchy of the dimensions or the behavior of the dimensionless ratios.

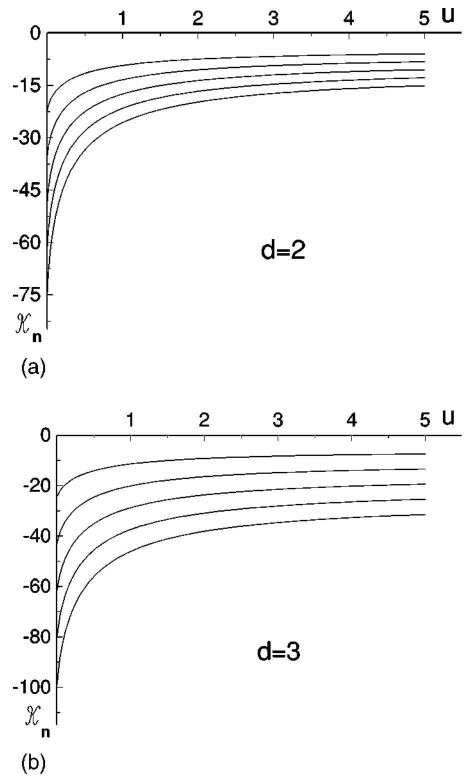

FIG. 2. Behavior of the quantity $\mathcal{K}_n(d,u)$ from Eq. (4.13) as a function of u for $n=2$, 3, 4, 5, and 6 (from above to below) for $d=2$ (left) and $d=3$ (right).

Consider the effects of the finite correlation time on the hierarchy of the anisotropic contributions. To this aim, consider the difference of the coefficients (4.1) for a fixed value of n and two neighboring subsequent values of l,

$$\Delta_{n,l+2}^{(2)}-\Delta_{nl}^{(2)}=\frac{2(2l+d)\ \mathcal{K}_n(d,u)}{(d-1)^2(d+2)^2(d+4)},$$

$$\mathcal{K}_n(d,u)\equiv\{2(d+4)\mathcal{A}+9(n-2)[2\mathcal{B}-(d+1)\mathcal{C}]\}. \tag{4.13}$$

(we recall that for a fixed n, all possible values of l are either even or odd, so that the subsequent values of l differ by 2). It is clear from Eq. (4.13) that the sign and the dependence on u of the whole expression is determined by the behavior of the function $\mathcal{K}_n(d,u)$.

In Fig. 2, we plot the quantity $\mathcal{K}_n(d,u)$ as a function of u for $n=2$, 4, 6, and 20 (from above to below) for $d=2$ [Fig. 2(a)] and $d=3$ [Fig. 2(b)]. The function is always negative for all the cases studied and increases monotonically with u. This behavior persists in the limit of large d, as follows from the asymptotical expression (4.7).

We thus conclude that the $O(\varepsilon^2)$ contribution in the exact dimension (3.10) "tries to cope" with the hierarchy, set by the $O(\varepsilon)$ term, for all values of n, l, d, and u; this effect is at its strongest for $u=0$ and weakens monotonically as u increases from 0 to ∞.

Now let us turn to the dimensionless ratios $\mathcal{R}_k$ in Eq. (4.12). From the discussion below Eq. (4.12) and asymptotic

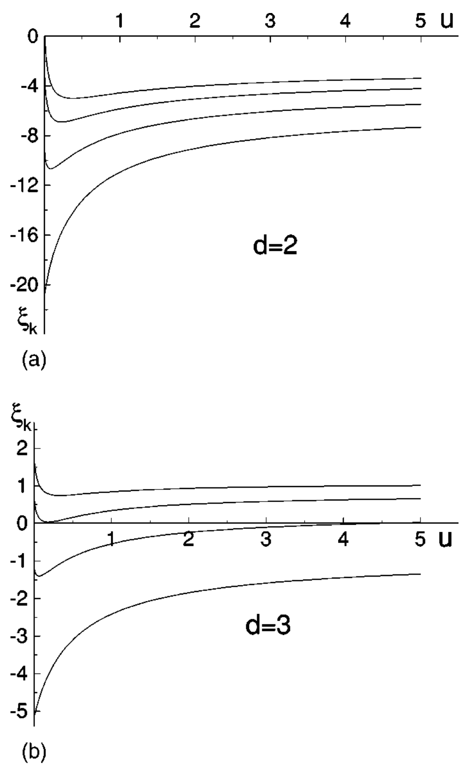

FIG. 3. Behavior of the quantity $\xi_k \equiv \Delta^{(2)}_{2k+1,1}/(2k+1)^3$ from Eq. (4.1) as a function of u for $k=1$, 2, 3, and 4 (from below to above) for $d=2$ (left) and $d=3$ (right), in the units of 10^{-3}.

representation (3.6) one obtains the powerlike inertial-range asymptotic expression $\mathcal{R}_k \propto (mr)^{\Delta_{2k+1,1}}$ with $\Delta_{2k+1,1}$ from Eq. (3.10) (we recall that in our model $\Delta_{2,0}=0$; see Sec. IV B). Due to the u independence of the first-order answer (3.11), the $O(\varepsilon)$ contribution in the exponent $\Delta_{2k+1,1}=\varepsilon(d+2-4k^2)/2(d+2)+O(\varepsilon^2)$ coincides with its analog for the Kraichnan model; see Ref. [5] for $k=1$ and Ref. [19] for general k. It completely determines the qualitative behavior of the quantities $\mathcal{R}_k$: for $k=1$ one has $\Delta_{3,1}>0$ and the skewness factor $\mathcal{R}_1$ decreases with mr, while for $k \geqslant 1$ one has $\Delta_{2k+1,1}<0$ and the higher-order ratios $\mathcal{R}_k$ increase for $mr \to 0$.

In Fig. 3, we show the behavior of the second-order correction $\Delta^{(2)}_{2k+1,1}$, obtained from the general formula [Eq. (4.1)] and divided by $(2k+1)^3$ for the even dimensions, as a function of u for $k=1$, 2, 3, and 4 (from below to above) for $d=2$ [Fig. 3(a)] and $d=3$ [Fig. 3(b)].

One can see that the effect of the $O(\varepsilon^2)$ correction on the inertial-range behavior of the ratios $\mathcal{R}_k$ is different for different d, k, and u. In two dimensions, the corrections are negative for all u and moderate k: the decay of the skewness factor $\mathcal{R}_1$ for $mr \to 0$ appears even slower than indicated by the $O(\varepsilon)$ approximation, while the growth of the ratios with $k \geqslant 2$ becomes faster.

In three dimensions, the correction is negative for $k=1$ so that the decay $\mathcal{R}_1$ for $mr \to 0$ is also slower than in the $O(\varepsilon)$ approximation. For $k=2$, the correction is negative for small u (so that the growth of the hyperskewness factor $\mathcal{R}_2$ for $mr \to 0$ is faster than in the first-order approximation), but it changes its sign for some finite value of u and the growth of $\mathcal{R}_2$ slows down. For $k \geqslant 3$, the corrections are negative for all u and the growth of the corresponding higher-order ratios $\mathcal{R}_k$ appears slower than predicted by the $O(\varepsilon)$ expression. One thus may conclude that for $d=3$, with the exception of the $k=2$ case, the effect of the second-order term is opposite to the tendency set by the first-order approximation.

For k large enough and any d, the behavior of the quantities $\Delta^{(2)}_{2k+1,1}$ becomes similar to that of the even dimensions $\Delta^{(2)}_{2k,0}$ discussed in Sec. IV B: they decrease monotonically as u increases, comparatively fast for small u (due to the singularity in the slope for $d=2$; see Sec. IV A) and rather slow when u becomes large enough. This follows from the fact that the l-independent contribution in the general expression $\Delta^{(2)}_{nl}$ behaves as $O(n^3)$ for $n \to \infty$, while its l-dependent contribution behaves only as $O(n)$; see Eq. (4.1).

We also note that for moderate k, the quantities $\Delta^{(2)}_{2k+1,1}$ show a nonmonotonous dependence on u in the region of small u and in this respect they also resemble the even dimensions; see Fig. 1 and the discussion in Sec. IV B.

V. CONCLUSION

We have applied the RG and OPE methods to a simple model of a passive scalar quantity advected by the synthetic Gaussian velocity field with a given self-similar covariance with finite correlation time. The structure functions of the scalar field exhibit inertial-range anomalous scaling behavior, as a consequence of the existence in the model of composite operators with negative scaling dimensions, identified with anomalous exponents.

For the special case of a local turnover exponent, the anomalous exponents are nonuniversal through the dependence on a dimensionless parameter u that has the meaning of the ratio of the velocity correlation time and the scalar turnover time. The universality reveals itself only in the second order of the ε expansion, and we have derived the exponents to order $O(\varepsilon^2)$, including anisotropic contributions.

It is shown that, for isotropic contributions, the qualitative effect of finite correlation time depends essentially on the order of the structure function n and the space dimensionality d. For moderate n and d, finite correlation time enhances the intermittency in comparison with both the limits: the rapid-change ($u=\infty$) and quenched ($u=0$) ones. The $O(\varepsilon^2)$ term shows a highly nontrivial behavior in the vicinity of the quenched limit: a rapid falloff when $u=0$ increases from zero, with infinite derivative at $u=0$ for $d=2$, with a pronounced minimum for $u \sim 1$. This irregularity shows that the time-independent advecting field can hardly be a reasonable approximation in studying more realistic models of passive advection by the velocity field with finite correlation time. The behavior near the opposite limit, $u=\infty$, is smooth in agreement with the existing simulation for a shell model [14].

The behavior changes remarkably when n and/or d become large enough: the correction to the limit $u=\infty$ due to finite correlation time is positive for all u (that is, the anomalous scaling is suppressed in comparison with the rapid-change case), it is maximal for $u=0$ and monotonically decreases to zero as u tends to infinity.

In the anisotropic sectors, the $O(\varepsilon^2)$ terms diminish the hierarchy revealed by the first-order terms for all values of the parameters n, l, and d; this effect is maximal at $u=0$ and decreases monotonically with $1/u$.

The effect of the $O(\varepsilon^2)$ corrections on the inertial-range behavior of the dimensionless ratios involving odd-order structure functions depends on d. For $d=2$ and moderate k these corrections are negative; the decay of the skewness factor $\mathcal{R}_1$ for $mr\to 0$ is slower while the growth of the higher-order ratios $\mathcal{R}_k$ with $k\geqslant 2$ is faster than indicated by the $O(\varepsilon)$ approximation by Refs. [5,18]. For $d=3$, the effect is, for most cases, opposite to the tendency set by the first-order approximation: both the decay of the skewness factor and the growth of the higher-order ratios become slower.

Our analysis has been confined within the region of small ε, where the results obtained within the ε expansion are internally consistent and undoubtedly reliable [we recall again that, although the leading terms of the anomalous exponents are of order $O(\varepsilon)$, the leading terms in which the effects of finite correlation time occur are of order $O(\varepsilon^2)$]. We do not discuss here the serious issue of validity of the ε expansions for finite $\varepsilon=O(1)$. One can think that, in our model, the natural region of validity of the ε expansion is restricted by the value $\varepsilon=1/2$, where the velocity field acquires negative critical dimension (along with all its powers) and new IR singularities, related to the well-known sweeping effects, occur in the diagrams; see the discussion in Ref. [18]. (It should be noted, however, that such singularities do not necessarily lead to a changeover in the inertial-range behavior, as shown in Ref. [18] for the special case of the structure function $\mathcal{S}_2$ for $u=0$.) On the other hand, $\varepsilon=1/2$ can be regarded as the upper bound of the range of validity of the model itself: the lack of Galilean covariance becomes a serious drawback of the synthetic Gaussian velocity ensemble when the sweeping effects become important. The next important step should be the analytical derivation of anomalous exponents of a passive scalar advected by the Galilean covariant velocity field; this work is now in progress.

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